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Resummed mass distribution for jets initiated by massive quarks

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This work is dedicated to the memory of Jiro Kodaira

Abstract

We resum the invariant mass distribution of jets initiated by massive quarks in next-to-leading logarithmic approximation and beyond in heuristic way. We find that the inclusion of mass terms, in the N-moment space, results in the universal factor $\delta_N(Q^2; m^2) = \exp[\int_0^1 dz \frac{z^{\frac{m^2}{Q^2}(N-1)}-1}{1-z}\{-\int_{m^2(1-z)^2}^{m^2(1-z)} \frac{dk_\perp^2}{k_\perp^2} A[\alpha_S(k_\perp^2)] - B[\alpha_S(m^2(1-z))] + D[\alpha_S(m^2(1-z)^2)]\}]$, taking into account dead-cone effects and soft radiation characteristic of massive charges. This factor multiplies the massless jet distribution function $J_N(Q^2)$. In the above equation the variable N is rescaled by the mass correction parameter $m^2/Q^2 \ll 1$ with respect to the standard massless case, being m the quark mass and Q the hard scale. The functions $A(\alpha_S)$ and $B(\alpha_S)$ appear with a minus sign suppressing collinear effects at very large $N \gtrsim Q^2/m^2$, as expected. In the same region, soft radiation not collinearly enhanced, characteristic of on-shell massive charges, makes its appearance with the function $D(\alpha_S)$. Phenomenological applications, such as the resummation of $b \to c l \nu$ decay spectra or the inclusion of beauty mass effects in $t \to bW$ decays, are briefly sketched.

1. Introduction

The structure of hadronic final states at high energy is largely determined by the infrared (soft and collinear) divergencies occurring in real contributions to QCD cross sections. The singularities cancel with those coming from virtual corrections in sufficiently inclusive observables, but leave often large residual logarithmic effects [1]. Soft singularities mean in physical terms a high probability of soft-gluon emission:

$$dP \sim \alpha \frac{dE}{E}, \tag{1}$$

where $\alpha \equiv \alpha_S$ with E the gluon energy and Q the hard scale ($\Lambda_{\mathrm{QCD}} \ll E \ll Q$, Λ_{QCD} being the hadronic scale). Because of such singularities, one has to think to a quark or a gluon as a "dressed" parton, which is always accompanied by a cloud of virtual soft gluons. In a scattering process, part of the cloud "detaches" and converts itself into real soft radiation, observed as low-energy hadrons. Collinear singularities in physical terms mean high probability of collinear configurations:

$$dP \sim \alpha \frac{d\theta^2}{\theta^2}, \tag{2}$$

where θ is the emission angle. Because of them, a massless quark or a gluon evolves into an ensemble of collinear partons, which show up in the detectors as a jet of hadrons. In the case of a *massless* parton, there is an overlap of soft and collinear regions

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producing the well-known (leading) double-logarithmic effects:

$$dP \sim \alpha \frac{dE}{E} \frac{d\theta^2}{\theta^2}. \tag{3}$$

Furthermore, there is *hard collinear* emission related to contributions of the form (2). In the case of a *massive* parton—which we assume to be observed in a reference frame where it is fast moving—two main changes occur in the QCD radiation. The first is the well-known "dead-cone" effect, according to which little radiation is emitted in the forward direction:

$$dP_m \sim \alpha \frac{d\theta^2}{\theta^2 + m^2/Q^2} \sim \alpha H\left(\theta^2 - \frac{m^2}{Q^2}\right) \frac{d\theta^2}{\theta^2}, \tag{4}$$

where H is the step function. The second effect is related to the fact that a massive parton radiates soft quanta *isotropically* in its *rest frame*:

$$dP_m \sim \alpha \frac{dE}{E}. \tag{5}$$

These quanta, being radiated in any space direction, cannot be ascribed in a natural way to any jet in the event. They are related to the "classical" chromo-electric field of a static color charge. By means of a Lorentz transformation, we obtain a fast-moving parton, which is now accompanied by a boosted Coulomb field, i.e. by soft radiation restricted to a forward cone of angular size $\theta \sim 1/\gamma_L \sim m/Q$. As a consequence, when we consider the structure of a jet initiated by a parton with a small mass, we expect *missing* collinear radiation compared to the massless case and *additional* soft radiation related to the boosted Coulomb field. A non-vanishing quark mass, $m \neq 0$, produces, as we are going to show in detail, a specific sub-structure, which is expected to be universal on the basis of physical intuition: small angle emissions only are involved, which can be ascribed to a specific jet in the event. In general, whenever logarithmically enhanced effects are encountered, one expects a factorized structure. As we are going to show in detail in this Letter, all these expectations turn out to be correct.

In the massless case, it is convenient to introduce the "jet function" $J(y; Q^2)$, which gives the probability that a massless parton produced in a hard process characterized by the scale Q fragments into a hadronic jet of mass squared [2–4]

$$m_X^2 = yQ^2. \tag{6}$$

In first order, its expression reads:

$$J\left(y; Q^2\right) = \delta(y) - A_1\alpha\left(\frac{\log y}{y}\right)_+ + B_1\alpha\left(\frac{1}{y}\right) + O\left(\alpha^2\right), \tag{7}$$

where A_1 and B_1 are coefficients whose explicit expressions will be given later and the plus distributions are defined as:

$$P(y)_+ \equiv \lim_{\epsilon\to 0^+}\left[\theta(y-\epsilon)P(y) - \delta(y-\epsilon)\int_\epsilon^1 P(y')\,dy'\right]. \tag{8}$$

Note that in the free limit, $\alpha \to 0$, the jet has a trivial structure and the mass distribution reduces to a spike corresponding to the (zero) parton mass. If $y \ll 1$, the coefficients of α in Eq. (7) become large, making a truncated perturbative expansion unreliable and asking for a resummation to all orders. The latter is systematically performed by going to N-space, i.e. by considering

$$J_N\left(Q^2\right) = \int_0^1 dy\,(1-y)^{N-1} J\left(y; Q^2\right). \tag{9}$$

The resummed expression reads [2–4]:

$$J_N\left(Q^2\right) = \exp\int_0^1 dy \frac{(1-y)^{N-1}-1}{y}\left\{\int_{Q^2y^2}^{Q^2y} \frac{dk_\perp^2}{k_\perp^2} A\left[\alpha\left(k_\perp^2\right)\right] + B\left[\alpha\left(Q^2 y\right)\right]\right\}. \tag{10}$$

The functions $A(\alpha)$ and $B(\alpha)$, generalizing the first-order expressions $A_1\alpha$ and $B_1\alpha$ respectively, describe small-angle *soft* and *hard* gluon emission off a massless parton respectively (see later).

In the massive case, one can define a *generalized* jet function $J(y; Q^2, m^2)$, where y is now defined as (cf. Eq. (6)):

$$y = \frac{m_X^2 - m^2}{Q^2 - m^2}. \tag{11}$$

To first order, the massive jet function reads:

$$J\big(y; Q^2, m^2\big) = \delta(y) + \alpha\left[(-A_1 \log y + B_1)\frac{\theta(y-r)}{y} + (-A_1 \log r + D_1)\frac{\theta(r-y)}{y}\right]_+ + O\big(\alpha^2\big), \tag{12}$$

where θ is the step function and D_1 is a coefficient which will be derived in the next section. We have defined the mass correction parameter

$$r \equiv \frac{m^2}{Q^2} \ll 1. \tag{13}$$

Note that r is quadratic in m and that we always assume that the quark mass is much smaller than the hard scale, $m \ll Q$, is in order to have fast-moving charges and to preserve a jet structure.[1]

According to Eq. (12), we can identify two different kinematical regions, which can be defined with logarithmic accuracy:

(1) *high* jet mass: $y \gg r$ or, equivalently, $m_X - m \gg m$. The quark mass m can be neglected and the collinear region produces $\log y$ terms;
(2) *low* jet mass: $y \ll r$ or, equivalently, $m_X - m \ll m$. The quark mass screens the collinear singularity and produces $\log r$ terms.

Mass effects are more easily looked at by considering the partially integrated jet rate:

$$R\big(y; Q^2, m^2\big) \equiv \int_0^y J\big(y'; Q^2, m^2\big)\, dy'. \tag{14}$$

In the high-mass region:

$$R(y) = 1 - \int_y^1 J\big(y'; Q^2, m^2\big)\, dy' = 1 - \frac{1}{2} A_1 \alpha \log^2 y + B_1 \alpha \log y \quad (y > r), \tag{15}$$

while in the low-mass region:

$$R(y) = 1 - A_1 \alpha \log y \log r + \frac{A_1}{2} \alpha \log^2 r + D_1 \alpha \log y + (B_1 - D_1)\alpha \log r \quad (y < r). \tag{16}$$

For $r \ll 1$ mass logarithms become large and need to be resummed to all orders. We will show that mass terms can be relegated into a factor which takes into account the effects discussed above. The jet function can be factorized in moment space as:

$$J_N\big(Q^2; m^2\big) = J_N\big(Q^2\big)\delta_N\big(Q^2; m^2\big), \tag{17}$$

where the mass-correction factor $\delta_N(Q^2; m^2)$ reads:

$$\delta_N\big(Q^2; m^2\big) = \exp \int_0^1 dy \frac{(1-y)^{r(N-1)} - 1}{y} \left\{ - \int_{m^2 y^2}^{m^2 y} \frac{dk_\perp^2}{k_\perp^2} A\big[\alpha\big(k_\perp^2\big)\big] - B\big[\alpha\big(m^2 y\big)\big] + D\big[\alpha\big(m^2 y^2\big)\big] \right\}. \tag{18}$$

The function $D(\alpha)$, generalizing the first-order term $D_1\alpha$, describes soft radiation not collinearly enhanced off a nearly on-shell massive quark (see later). We derive our main result, Eq. (18), with some rigor in next-to-leading logarithmic (NLL) approximation and we conjecture its general validity beyond NLL. Eq. (18) has a simple physical interpretation: the parameter $N - 1$ is multiplied by r on the r.h.s., implying that mass effects become "visible" only for large

$$N \gtrsim \frac{Q^2}{m^2} \gg 1. \tag{19}$$

In this case, there is enough resolution to "see" the quark mass, which tends to suppress the collinear effects, related to the A and B terms. At the same time, soft radiation not collinearly enhanced, described by the function D and characteristic of massive partons, does appear. Let us also note that, since the jet mass is an infrared (i.e. soft and collinear) safe quantity, $\delta_N = 1$ for $r = 0$.

For clarity's sake, we derive the general formula (18) in Section 2 for a specific physical process first, namely the radiative decay

$$B \to X_s + \gamma \tag{20}$$

[1] We are not interested to the case in which the quark mass is close to the hard scale, $m \approx O(Q)$, as this case corresponds to a non-relativistic motion of the color charges which does not give rise to jets.

by keeping $m_s \neq 0$.[2] In Section 3 we generalize the results of the previous section to the case of a process involving one or more jets initiated by quarks with a small mass compared to the hard scale. Finally in Section 4 we present the conclusions of our study and we discuss natural applications and developments.

2. Mass distribution in $B \to X_s + \gamma$ for $m_s \neq 0$

In this section we compute the invariant hadron mass distribution in the decay (20), having the fixed-order expansion:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dy} = \delta(y) + \frac{1}{\Gamma}\frac{d\Gamma^{(1)}}{dy} + \frac{1}{\Gamma}\frac{d\Gamma^{(2)}}{dy} + \cdots, \tag{21}$$

where $1/\Gamma\, d\Gamma^{(n)}/dy = O(\alpha^n)$ and Γ is the corrected inclusive width. The massless case has been considered for example in Ref. [5].

2.1. Single-gluon distribution

In order to include the quark-mass corrections to Eq. (21), let us start from the eikonal current containing explicitly the mass terms. According to standard factorization formulae, the soft terms in the amplitude for

$$b \to s + g + \gamma \tag{22}$$

are factorized by the eikonal current

$$J^\mu(k) = ig\left(\mathbf{T_b}\frac{p_b^\mu}{p_b \cdot k} - \mathbf{T_s}\frac{p_s^\mu}{p_s \cdot k}\right), \tag{23}$$

where g is the coupling in the QCD Lagrangian, k^μ is the soft-gluon momentum and $\mathbf{T_b}$ and $\mathbf{T_s}$ are the color generators for the beauty and the strange quark respectively [6]. Since the photon—in general the probe—carries no color, we have color conservation along the quark line:

$$\mathbf{T_b} = \mathbf{T_s}. \tag{24}$$

The square of the eikonal current reads[3]:

$$\begin{aligned} -g_{\mu\nu} J^\mu(k) J^\nu(k)^* &= 4\pi\alpha C_F\left[\frac{2(p_b \cdot p_s)}{(p_b \cdot k)(p_s \cdot k)} - \frac{m_b^2}{(p_b \cdot k)^2} - \frac{m_s^2}{(p_s \cdot k)^2}\right] \\ &= 4\pi\alpha\frac{C_F}{E_g^2}\left\{\frac{1+r}{1-r}\frac{1}{t + r/(1-r)} - 1 - \frac{r}{(1-r)^2}\frac{1}{[t + r/(1-r)]^2}\right\}, \end{aligned} \tag{25}$$

where we have used the fact that $\mathbf{T_b^2} = \mathbf{T_s^2} = \mathbf{T_b} \cdot \mathbf{T_s} = C_F = 4/3$ with C_F the usual Casimir of the fundamental representation of $SU(3)$, E_g is the soft-gluon energy and t is a unitary angular variable

$$t \equiv \frac{1 - \cos\vartheta}{2} \tag{26}$$

with ϑ the emission angle of the gluon with respect to the strange quark (in the soft limit there is no recoil and the direction of the strange quark is not modified by gluon emission). The mass-correction parameter is given in this case by

$$r \equiv \frac{m_s^2}{m_b^2}. \tag{27}$$

Note that in the massless limit for the strange quark, the last term on the r.h.s. of Eq. (25) identically vanishes. The ordinary quark velocity and the Lorentz factor are given respectively by:

$$v \equiv \frac{p_s}{E_s} = \frac{1-r}{1+r}; \qquad \gamma \equiv \frac{E_s}{m_s} = \frac{1+r}{2\sqrt{r}}. \tag{28}$$

[2] Let us note that the inclusion of strange mass effects is a rather academical problem because even with a large value of the constituent mass $m_s = 0.5$ GeV, the correction parameter is very small: $r = (m_s/m_b)^2 \approx 10^{-2}$. Furthermore, strange mass effects become visible for $x_\gamma \equiv 2E_\gamma/m_B = 1 - m_{X_s}^2/m_B^2 \gtrsim 0.99$, where resonance effects due to K, K^* peaks, etc.—intrinsically beyond perturbative QCD—are substantial.

[3] For simplicity's sake, we compute the square in Feynman (i.e. covariant) gauge, but the result is gauge invariant because the eikonal current is conserved: $k_\mu J^\mu(k) = 0$.

By taking the ultra-relativistic limit $r \ll 1$, we obtain:

$$-g_{\mu\nu} J^{\mu}(k) J^{\nu}(k)^* \cong 4\pi\alpha \frac{C_F}{E_g^2}\left[\frac{1}{t+r} - 1 - \frac{r}{(t+r)^2}\right]. \tag{29}$$

We now introduce the approximations

$$\frac{1}{t+r} \simeq \frac{\theta(t-r)}{t}, \qquad \frac{1}{(t+r)^2} \simeq \theta(t-r)\frac{1}{t^2}, \tag{30}$$

to extract the leading behavior of the matrix element squared for $r \ll 1$.[4] We then obtain:

$$-g_{\mu\nu} J^{\mu}(k) J^{\nu}(k)^* \simeq 4\pi\alpha \frac{C_F}{E_g^2}\left[\frac{\theta(t-r)}{t} - 1 - \frac{r\theta(t-r)}{t^2}\right] \simeq 4\pi\alpha \frac{C_F}{E_g^2}\theta(t-r)\left[\frac{1}{t} - 1 - \frac{r}{t^2}\right]. \tag{32}$$

On the last member we have multiplied the term independent from t (related to an isotropic soft-gluon emission off the beauty quark at rest) by the θ function, as this only introduces $O(r)$ terms and allows a factorization of this "dynamical" constraint.

The approximations (30) actually constitute the dead-cone approximation, which is the following limitation on lower emission angles of the gluon with respect to the strange quark:

$$\vartheta > \vartheta_{\min} \equiv \frac{m_s}{E_s^{(0)}}, \tag{33}$$

where $E_s^{(0)}$ is the strange quark energy in lowest order, which can be identified with the jet energy in the soft limit

$$E_{Xs} \simeq E_s^{(0)} = \frac{m_b}{2}(1+r), \tag{34}$$

and m_b is the beauty (pole) mass. We then obtain for the angular variable the corresponding limitation

$$t > t_{\min} = \frac{1-\cos\vartheta_{\min}}{2} \simeq \left(\frac{\vartheta_{\min}}{2}\right)^2 \simeq r. \tag{35}$$

An analogous simplification can be made for the kinematical constraint of given jet mass, i.e. of fixed (cf. Eq. (11))

$$y \equiv \frac{m_{Xs}^2 - m_s^2}{m_b^2 - m_s^2}, \tag{36}$$

with $m_{Xs}^2 \equiv (p_s + p_g)^2$. Let us note that y is a unitary variable equal to zero in the Born kinematics point $m_{Xs} = m_s$. For a jet containing one soft gluon:

$$y = (1-r)\omega\left[t + \frac{r}{1-r}\right] \cong \omega(t+r) \simeq \omega t \quad \text{for } t > r, \tag{37}$$

where

$$\omega \equiv \frac{2E_g}{m_b(1-r)} \tag{38}$$

is the normalized gluon energy.

To sum up: the soft-limit mass distribution for a jet initiated by a massive quark can be obtained from the corresponding massless formula by simply adding the term explicitly proportional to m_s^2 and imposing the dead-cone effect:

$$\frac{1}{\Gamma}\frac{d\Gamma^{(1)}}{dy}\bigg|_{\text{soft}} = \int_0^1 d\omega \int_r^1 dt \left[\frac{A_1\alpha}{\omega t} + \frac{D_1\alpha}{\omega} + \frac{D_1 r\alpha}{\omega t^2}\right]\left[\delta(y-\omega t) - \delta(y)\right]. \tag{39}$$

The first-order coefficients read [5]:

$$A_1 = \frac{C_F}{\pi}; \qquad D_1 = -\frac{C_F}{\pi}. \tag{40}$$

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[4] Eqs. (30) are intended to hold in an integral way:

$$\int_0^1 \frac{dt}{t+r} = \int_r^1 \frac{dt}{t} + O(r); \qquad \int_0^1 dt \frac{1}{(t+r)^2} = \int_r^1 dt \frac{1}{t^2} + O(r). \tag{31}$$

The first term proportional to D_1 on the r.h.s. of Eq. (39) is related to soft emission off the initial beauty quark, in its rest frame, while the second one is related to soft emission off the fast-moving strange quark.

Eq. (39) misses the contribution from hard collinear gluon emission. We assume that the latter can be obtained from the massless one [5] by simply imposing the dead-cone effect. The complete one-gluon distribution therefore reads:

$$\frac{1}{\Gamma}\frac{d\Gamma^{(1)}}{dy} = \int_0^1 d\omega \int_r^1 dt \left[\frac{A_1\alpha}{\omega t} + \frac{D_1\alpha}{\omega} + \frac{D_1\alpha r}{\omega t^2} + \frac{B_1\alpha}{t}\right]\left[\delta(y-\omega t) - \delta(y)\right], \tag{41}$$

where

$$B_1 = -\frac{3}{4}\frac{C_F}{\pi}. \tag{42}$$

A check of our assumption is provided by the comparison with the fixed order Feynman diagram computation of the spectrum given in Section 2.2.[5] The event fraction is given by:

$$R(y) \equiv \int_0^y \frac{1}{\Gamma}\frac{d\Gamma}{dy'}dy' = 1 + R^{(1)}(y) + \cdots, \tag{43}$$

where:

$$R^{(1)}(y) = -\int_0^1 d\omega \int_r^1 dt \left[\frac{A_1\alpha}{\omega t} + \frac{D_1\alpha}{\omega} + \frac{D_1\alpha r}{\omega t^2} + \frac{B_1\alpha}{t}\right]\theta(\omega t - y). \tag{44}$$

Note that the formula above has the correct end-point value: $R(1) = 1$.

2.1.1. Leading order

Let us start by considering the leading-order term proportional to the coefficient A_1 in Eq. (41). To include leading higher-order effects, we replace the tree-level coupling with the running coupling evaluated at the gluon transverse momentum squared [7] (see later):

$$\alpha \to \alpha\left(k_\perp^2\right), \tag{45}$$

where the gluon transverse momentum squared is defined as [2]:

$$k_\perp^2 \equiv (1-r)^2 m_b^2 \omega^2 t \simeq E_g^2 \vartheta^2 \quad \text{for } \vartheta \ll 1. \tag{46}$$

By integrating the leading-order term on the r.h.s. of Eq. (41) and expressing the result in terms of the gluon transverse momentum, we obtain:

$$\int_0^1 \frac{d\omega}{\omega}\int_r^1 \frac{dt}{t} A_1\alpha\left(k_\perp^2\right)\delta(y-\omega t) = \frac{1}{y}\int_{m_b^2 y^2}^{m_b^2 y \min[1, y/r]} \frac{dk_\perp^2}{k_\perp^2} A_1\alpha\left(k_\perp^2\right) \simeq \frac{1}{y}\int_{m_b^2 y^2}^{m_b^2 y^2/(y+r)} \frac{dk_\perp^2}{k_\perp^2} A_1\alpha\left(k_\perp^2\right), \tag{47}$$

where on the last member a smooth interpolation between the two regions specified by the minimum has been considered according to the approximation[6] $\max[y, r] \approx y + r$. Eq. (47) reduces in the frozen coupling case to $-A_1\alpha/y \log(y+r)$. As anticipated in the introduction, we have therefore two different regions:

(1) *high jet mass*—compared to quark mass m_s:

$$y \gg r. \tag{48}$$

Up to the logarithmic accuracy we are interested in, one can extend this region up to $y > r$. Eq. (47) reduces to the massless case:

$$\frac{1}{y}\int_{m_b^2 y^2}^{m_b^2 y} \frac{dk_\perp^2}{k_\perp^2} A_1\alpha\left(k_\perp^2\right). \tag{49}$$

5 A more rigorous and direct derivation can be obtained by using the massive splitting functions (see for example [6]).

6 In general, step approximations are more convenient to obtain simple analytical results while continuous functions are best suited for numerical purposes.

In this region, the jet mass is so large that, at the logarithmic level, no effect is left of $m_s \neq 0$. To first order, Eq. (49) becomes $-A_1\alpha/y \log y$;

(2) *low jet mass*:

$$y \ll r. \tag{50}$$

Analogously to the previous case, one can actually extend this region up to $y < r$. Eq. (47) specializes to:

$$\frac{1}{y}\int_{m_b^2 y^2}^{m_b^2 y^2/r} \frac{dk_\perp^2}{k_\perp^2} A_1\alpha\big(k_\perp^2\big). \tag{51}$$

It is worth noting that the effect of a non-zero mass, $m_s \neq 0$, is a restriction on the *upper* gluon transverse momenta. To first order, Eq. (51) becomes $-A_1\alpha/y \log r$. In this low jet-mass region, in agreement with physical intuition, the s quark mass m_s completely screens the collinear singularity and produces a logarithm of the quark mass m_s normalized to the relevant hard scale m_b.

2.1.2. *Subleading effects*

Let us discuss in this section the derivation of the sub-leading effects in Eq. (41):

(1) the first term proportional to D_1, related to soft emission off the initial heavy parton at rest, reads:

$$\int_0^1 \frac{d\omega}{\omega}\int_r^1 dt\, D_1\alpha\big(k_\perp^2\big)\delta(y-\omega t) \simeq D_1\alpha\big(m_b^2 y^2\big)\frac{1}{y}. \tag{52}$$

This term is not modified with respect to the massless case, again as expected on the basis of physical intuition;

(2) the second term proportional to D_1, related to soft emission off the massive and fast-moving strange quark, reads:

$$\int_0^1 \frac{d\omega}{\omega}\int_r^1 \frac{dt}{t^2} r D_1\alpha\big(k_\perp^2\big)\delta(y-\omega t) \simeq \alpha\left(\frac{m_b^2 y^2}{y+r}\right) D_1\left(\frac{1}{y}-\frac{1}{y+r}\right) \simeq \alpha\left(\frac{m_b^2 y^2}{r}\right) D_1 \frac{\theta(r-y)}{y}. \tag{53}$$

This term therefore vanishes in the high jet-mass region $y \gg r$, where we recover the massless case (in which this term was absent from the very beginning);

(3) the term proportional to B_1, related to hard collinear emission off the strange quark, reads:

$$\int_0^1 d\omega\int_r^1 \frac{dt}{t} B_1\alpha\big(k_\perp^2\big)\delta(y-\omega t) \simeq B_1\alpha\left(\frac{m_b^2 y^2}{y+r}\right)\frac{1}{y+r} \simeq B_1\alpha\big(m_b^2 y\big)\frac{\theta(y-r)}{y}, \tag{54}$$

where we have neglected small terms beyond the logarithmic accuracy. This term is regulated by the non-vanishing strange mass and, as expected, we re-obtain the massless case with the infrared cutoff r on y.

By collecting the various contributions computed in the previous sections, we obtain for the $O(\alpha)$ distribution:

$$\frac{1}{\Gamma}\frac{d\Gamma^{(1)}}{dy} = -A_1\alpha\frac{\log(y+r)}{y} + 2D_1\alpha\frac{1}{y} + (B_1 - D_1)\alpha\frac{1}{y+r}. \tag{55}$$

Within logarithmic accuracy, we can make a sharp approximation to obtain:

$$\frac{1}{\Gamma}\frac{d\Gamma^{(1)}}{dy} = \begin{cases} -A_1\alpha \log r\,\frac{1}{y} + 2D_1\alpha\frac{1}{y}, & \text{for } y < r; \\ -A_1\alpha\frac{\log y}{y} + (B_1 + D_1)\alpha\frac{1}{y}, & \text{for } y > r. \end{cases} \tag{56}$$

The main effects of virtual corrections are included, as usual, by replacing the above functions with corresponding plus distributions. The event fraction reads in the high-mass region:

$$R(y) = 1 - \int_y^1 \frac{1}{\Gamma}\frac{d\Gamma}{dy'}dy' = 1 - \frac{1}{2}A_1\alpha\log^2 y + (B_1 + D_1)\alpha\log y \quad (y > r). \tag{57}$$

By imposing that the event fraction is continuous across $y = r$,[7]

$$R(y \to r^-) = R(y \to r^+) = 1 - \frac{1}{2} A_1 \alpha \log^2 r + (B_1 + D_1)\alpha \log r, \tag{58}$$

we obtain the $\log r$ terms in the low-mass region:

$$R(y) = 1 - A_1 \alpha \log y \log r + \frac{A_1}{2} \alpha \log^2 r + 2 D_1 \alpha \log y + (B_1 - D_1)\alpha \log r \quad (y < r). \tag{59}$$

2.2. Check with fixed-order computation

We compare in this section our Eq. (43) with the $O(\alpha)$ Feynman diagram computation of Ref. [8], where the decay spectrum has been computed by retaining the non-vanishing strange-quark mass. The relevant contribution is the one coming from the magnetic penguin operator

$$O_7 = \frac{e}{16\pi^2} m_{b,\overline{\mathrm{MS}}}(\mu_b) \bar{s} \sigma_{\mu\nu} b F^{\mu\nu}, \tag{60}$$

$m_{b,\overline{\mathrm{MS}}}(\mu_b)$ is the b mass in the $\overline{\mathrm{MS}}$ scheme and $\mu_b = O(m_b)$ is the renormalization scale. By omitting non-logarithmic terms, the fixed-order (fo) distribution in Eq. (30) of the first reference in [8] reads:

$$\frac{1}{\Gamma} \frac{d\Gamma^{(1)}}{dy}\bigg|_{\mathrm{fo}} = -A_1 \alpha \frac{\log(y+r)}{y} + 2 D_1 \alpha \frac{1}{y} + (B_1 - D_1)\alpha \frac{1}{y+r}. \tag{61}$$

The above formula exactly coincides with the first-order expression in Eq. (55) derived in the previous section. This comparison confirms the validity of the inclusion of the coefficients of leading and subleading contributions as given in Eq. (41).

2.3. N-space

In order to resum the distribution to all orders in α, one has to transform it to moment N-space. The invariant mass distribution in N-space is defined as:

$$\frac{\Gamma_N^{(1)}}{\Gamma} = \int_0^1 dy\, (1-y)^{N-1} \frac{1}{\Gamma} \frac{d\Gamma^{(1)}}{dy}. \tag{62}$$

The transform to N moment space indeed allows us to obtain an expression more suitable for the resummation of the logarithmic terms. As discussed in [9,10] for the case of the jet mass event variable, the Mellin transform yields to a factorized expression of the phase space.[8] That gives rise to expressions for the jet mass distribution which are accurate at the next-to-next-to-leading order [9,10]. The resulting expressions can be resummed in straightforward way. We will follow here the same procedure in order to resum the mass logarithms in Eq. (41). The exponent in the resummed expression reproduces the usual structure, common to several resummed variables, as the one shown in Refs. [2,9,10]. In practice, one exponentiates the one-gluon distribution in N-space to account for multiple emission:

$$\frac{\Gamma_N}{\Gamma} = 1 + \frac{\Gamma_N^{(1)}}{\Gamma} + \cdots \Rightarrow \exp\left[\frac{\Gamma_N^{(1)}}{\Gamma}\right]. \tag{63}$$

Gluon branching, i.e. secondary emission, is taken into account by evaluating the running coupling at the gluon transverse momentum squared [7]:

$$\alpha \to \alpha(k_\perp^2). \tag{64}$$

The "effective" one-gluon distribution therefore reads:

$$\frac{\Gamma_N^{(1)}}{\Gamma} = \int_0^1 dy\, (1-y)^{N-1} \int_0^1 d\omega \int_r^1 dt \left[\alpha(k_\perp^2)\frac{A_1}{\omega t} + \alpha(k_\perp^2)\frac{D_1}{\omega} + \alpha(k_\perp^2)\frac{D_1 r}{\omega t^2} + \alpha(k_\perp^2)\frac{B_1}{t}\right]\left[\delta(y - \omega t) - \delta(y)\right]. \tag{65}$$

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[7] That is equivalent to using the plus distributions.

[8] The factorization of matrix elements follows instead from the dynamics of QCD.

The virtual contributions can be "transferred" to the moment kernel as usual:

$$\frac{\Gamma_N^{(1)}}{\Gamma} = \int_0^1 dy\,\big[(1-y)^{N-1}-1\big]\int_0^1 d\omega \int_r^1 dt\left[\alpha\big(k_\perp^2\big)\frac{A_1}{\omega t}+\alpha\big(k_\perp^2\big)\frac{D_1}{\omega}+\alpha\big(k_\perp^2\big)\frac{D_1 r}{\omega t^2}+\alpha\big(k_\perp^2\big)\frac{B_1}{t}\right]\delta(y-\omega t). \tag{66}$$

In the following sections we perform the integrations of the various terms above.

2.4. Resummation

By putting all the pieces together, we obtain the jet-mass distribution for the heavy flavor decay in N-space:

$$\frac{\Gamma_N}{\Gamma} \simeq \exp\int_0^1 \frac{dy}{y}\big[(1-y)^{N-1}-1\big]\left\{\int_{m_b^2 y^2}^{m_b^2 y\min[1,y/r]} \frac{dk_\perp^2}{k_\perp^2} A_1\alpha\big(k_\perp^2\big)+\theta(y-r)B_1\alpha\big(m_b^2 y\big)+D_1\alpha\big(m_b^2 y^2\big)\right.$$
$$\left.+\,\theta(r-y)D_1\alpha\big(m_b^2 y^2/r\big)\right\}. \tag{67}$$

A consistent next-to-leading logarithmic (NLL) resummation can be realized as in [2]. The first-order coefficients A_1, D_1 and B_1 have been explicitly computed in the previous section. The remaining NLL terms are related to two-loop effects. In order to take them into account, in practice one has to include:

(1) the two-loop correction ($\propto \beta_1$) in the QCD coupling α in the leading term $A_1\alpha$;
(2) the two-loop correction to the first-order term $A_1\alpha$ by means of the replacement

$$A_1\alpha \to A_1\alpha + A_2\alpha^2. \tag{68}$$

It is pretty well established that the latter function is universal—in the framework of effective theories, it is the well-known cusp anomalous dimension [11]

$$\Gamma_{\text{cusp}}(\alpha) = \Gamma_{\text{cusp}}^{(1)}\alpha + \Gamma_{\text{cusp}}^{(2)}\alpha^2 + \cdots. \tag{69}$$

What about resummation in higher orders, i.e. NNLL and beyond? The coefficients A_1, A_2, B_1 and D_1 represent the lowest order terms of the functions[9]:

$$A(\alpha) = \sum_{n=1}^{\infty} A_n\alpha^n; \qquad B(\alpha) = \sum_{n=1}^{\infty} B_n\alpha^n; \qquad D(\alpha) = \sum_{n=1}^{\infty} D_n\alpha^n. \tag{70}$$

The functions $A(\alpha)$ and $B(\alpha)$ are related to small-angle emission only and therefore represent universal intra-jet properties, as confirmed by explicit higher-order computations [13]. The inclusion of the terms proportional to β_2, A_3 and B_2 is therefore rather safe—all these coefficients refer to universal ultraviolet or intra-jet quantities [14].

On the contrary, the function $D(\alpha)$, being related to soft emission at large angle, is in general a process-dependent inter-jet quantity. In the framework of fragmentation functions, the function

$$D^{(f)}(\alpha) = D_1^{(f)}\alpha + D_2^{(f)}\alpha^2 + \cdots \tag{71}$$

has been originally introduced in [4], where it was called $H(\alpha)$; the first-order coefficient H_1 was also explicitly computed—see the discussion in [15]. The D-function entering heavy flavor decays

$$D^{(h)}(\alpha) = D_1^{(h)}\alpha + D_2^{(h)}\alpha^2 + \cdots \tag{72}$$

has however been shown to coincide with the former one to all orders [16]:

$$D^{(f)}(\alpha) = D^{(h)}(\alpha). \tag{73}$$

The function $D^{(i)}(\alpha)$ with $i = f, h$ refers basically to soft radiation emitted by a heavy parton with a small virtuality. In heavy flavor decays such as

$$b \to X_s + \gamma, \tag{74}$$

9 A compilation of the known coefficients A_i, B_i and D_i in our normalization, with references to the original papers, can be found in [12].

one considers a heavy flavor in its rest frame and looks at the final invariant mass distribution, while in heavy flavor fragmentation processes such as

$$Z^0 \to b + \bar{b}, \tag{75}$$

one has a fast-moving beauty quark and looks at its energy. Since the function $D(\alpha)$ is the same in both cases, that seems to imply that the kinematical observable is irrelevant as long as soft-enhanced quantities are concerned. The only thing that matters is that of having a heavy quark close to its on-shell point. The first order coefficients of the above functions $D^{(f)}(\alpha)$ and $D^{(h)}(\alpha)$ coincide with our D_1. As discussed in the introduction, our $D(\alpha)$ is related to massive partons but it represents small-angle intra-jet corrections. On the basis of physical intuition, we therefore conjecture that this coincidence extends to higher orders, i.e. that our $D(\alpha)$ coincides with $D^{(f)}(\alpha)$ or $D^{(h)}(\alpha)$. From the above discussion it is clear that the inclusion of all the NNLL terms is not as rigorous as in the NLL case, the weakest point being the inclusion of the D_2 term. An explicit check of our guess can be obtained by comparing our expanded resummation formula (given at the end of the Letter) with an explicit (massive) two-loop computation, as soon as the latter becomes available.

Beyond NNLL approximation, only the coefficient B_3 is known analytically [13]. We assume that the resummation formula keeps the same structure and that by including (unknown) higher-order terms to our expansions,

$$A_1\alpha + A_2\alpha^2 + A_3\alpha^3 \to A(\alpha), \qquad B_1\alpha + B_2\alpha^2 \to B(\alpha), \qquad D_1\alpha + D_2\alpha^2 \to D(\alpha), \tag{76}$$

we obtain the decay spectrum formally resummed to all orders:

$$\frac{\Gamma_N}{\Gamma} \simeq \exp\int_0^1 \frac{dy}{y}\left[(1-y)^{N-1}-1\right]\left\{\int_{m_b^2 y^2}^{m_b^2 y \min[1,y/r]} \frac{dk_\perp^2}{k_\perp^2} A\left[\alpha\left(k_\perp^2\right)\right] + \theta(y-r) B\left[\alpha\left(m_b^2 y\right)\right] + D\left[\alpha\left(m_b^2 y^2\right)\right]\right.$$
$$\left. + \theta(r-y) D\left[\alpha\left(m_b^2 y^2/r\right)\right]\right\}. \tag{77}$$

In the limit $r \to 0$, we recover the well-known massless result. It is remarkable that the single logarithmic terms B and D have θ-functions of opposite arguments and therefore exclude each other: either a soft contribution is present, for a small jet-mass, or a collinear one is present, for a high jet mass. We now explicitly factor out the massless contribution in order to obtain the mass-correction factor δ_N given in the introduction:

$$\frac{\Gamma_N}{\Gamma} = \exp\int_0^1 \frac{dy}{y}\left[(1-y)^{N-1}-1\right]\left\{\int_{m_b^2 y^2}^{m_b^2 y} \frac{dk_\perp^2}{k_\perp^2} A\left[\alpha\left(k_\perp^2\right)\right] + B\left[\alpha\left(m_b^2 y\right)\right] + D\left[\alpha\left(m_b^2 y^2\right)\right]\right.$$
$$\left. + \theta(r-y)\left[-\int_{m_b^2 y^2/r}^{m_b^2 y} \frac{dk_\perp^2}{k_\perp^2} A\left[\alpha\left(k_\perp^2\right)\right] - B\left[\alpha\left(m_b^2 y\right)\right] + D\left[\alpha\left(m_b^2 y^2/r\right)\right]\right]\right\}. \tag{78}$$

In the massive case, one has therefore the additional factor[10]:

$$\delta_N\left(m_b^2; m_s^2\right) = \exp\int_0^r \frac{dy}{y}\left[(1-y)^{N-1}-1\right]\left\{-\int_{m_b^2 y^2/r}^{m_b^2 y} \frac{dk_\perp^2}{k_\perp^2} A\left[\alpha\left(k_\perp^2\right)\right] - B\left[\alpha\left(m_b^2 y\right)\right] + D\left[\alpha\left(m_b^2 y^2/r\right)\right]\right\}. \tag{80}$$

The integral above can be transformed back to unitary range by means of the rescaling $v = y/r$, which gives:

$$\delta_N\left(m_b^2; m_s^2\right) = \exp\int_0^1 \frac{dv}{v}\left[(1-rv)^{N-1}-1\right]\left\{-\int_{m_s^2 v^2}^{m_s^2 v} \frac{dk_\perp^2}{k_\perp^2} A\left[\alpha\left(k_\perp^2\right)\right] - B\left[\alpha\left(m_s^2 v\right)\right] + D\left[\alpha\left(m_s^2 v^2\right)\right]\right\}. \tag{81}$$

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[10] A compact derivation of the correction factor δ_N is obtained by splitting the angular integral in all the terms on the r.h.s. of Eq. (66) but the term proportional to $D_1 r$, as:

$$\int_r^1 dt = \int_0^1 dt - \int_0^r dt. \tag{79}$$

The first integral on the r.h.s. represents the massless case while the second one the mass corrections. The unitary range in the latter is restored by setting $t' = t/r$.

It is remarkable that, as a consequence of the rescaling, the hard scale Q (appearing in the limits of the transverse momentum integral as well as in the argument of the coupling in the single-logarithmic terms) changes from m_b to m_s. Let us also note that, in the last equation, the hard scale m_b only enters through the variable r inside the moment kernel.

To express the correction factor as a standard Mellin transform, we use the relation:

$$(1-ry)^{N-1}-1=(1-y)^{r(N-1)}-1+O\left(\frac{1}{Nr}\right). \tag{82}$$

Eq. (82) can be easily shown to be valid at the next-to-next-to-leading $\log N$ level, by using the relation [17]

$$(1-y)^{N-1}-1\simeq-\theta\left(y-\frac{1}{n}\right)+\frac{z(2)}{2}\left[\frac{1}{n}\delta\left(y-\frac{1}{n}\right)-\frac{1}{n^2}\delta'\left(y-\frac{1}{n}\right)\right]+O\left[N^3LO,\frac{1}{N}\right], \tag{83}$$

where $n\equiv Ne^{\gamma_E}$, $\gamma_E\equiv\lim_{k\to\infty}[\sum_{j=1}^{k}1/j-\log k]=0.577216\ldots$ is the Euler constant, $z(a)\equiv\sum_{n=1}^{\infty}1/n^a$ is the Riemann zeta function with $z(2)=\pi^2/6=1.64493\ldots$. The final expression for the correction factor in N-space therefore reads:

$$\delta_N\left(m_b^2;m_s^2\right)=\exp\int_0^1\frac{dv}{v}\left[(1-v)^{r(N-1)}-1\right]\left\{-\int_{m_s^2v^2}^{m_s^2v}\frac{dk_\perp^2}{k_\perp^2}A\left[\alpha\left(k_\perp^2\right)\right]-B\left[\alpha\left(m_s^2v\right)\right]+D\left[\alpha\left(m_s^2v^2\right)\right]\right\}. \tag{84}$$

3. General case

This is the central section of the Letter and contains general results about threshold resummation in processes with jets initiated by partons with a small mass compared to the hard scale.

3.1. Mass effects in a jet

We generalize in this section the resummation formula obtained in the previous section for the radiative b decay by simply noting that any reference to the particular process disappears in the correction factor δ_N in Eq. (84). Therefore we simply replace the beauty mass m_b with the hard scale Q of the general process and m_s with the mass m of the quark triggering the jet under consideration:

$$\delta_N\left(Q^2;m^2\right)=\exp\int_0^1dz\frac{z^{r(N-1)}-1}{1-z}\left\{-\int_{m^2(1-z)^2}^{m^2(1-z)}\frac{dk_\perp^2}{k_\perp^2}A\left[\alpha\left(k_\perp^2\right)\right]-B\left[\alpha\left(m^2(1-z)\right)\right]+D\left[\alpha\left(m^2(1-z)^2\right)\right]\right\}, \tag{85}$$

where r is now defined in Eq. (13). A check of our generalization is provided by the comparison with the full $O(\alpha)$ computation of the DIS cross section with a massive quark in the final state (see next section).

As anticipated in the introduction, the jet mass distribution is an infrared safe quantity, i.e. it has a finite limit for vanishing quark mass, $m\to0$. Our results are in agreement with this general fact in the following way: for $r\neq0$, the distribution contains mass logarithms $\sim\log r$ which are not power suppressed, but have support in the region of power-suppressed size $y<r$.[11] The $O(\alpha)$ term in the correction factor in physical space $\delta(y;Q^2,m^2)$ can be obtained by subtracting out line 2 (the massless distribution) from line 1 (the massive distribution) in Eq. (56):

$$\delta\left(y;Q^2,m^2\right)=\delta(y)-A_1\alpha\left[\theta(r-y)\frac{\log r/y}{y}\right]_+ +(D_1-B_1)\alpha\left[\frac{\theta(r-y)}{y}\right]_+ +O\left(\alpha^2\right). \tag{87}$$

Let us note that δ actually vanish for $y>r$ (where $r\ll1$).

3.2. Check with DIS with massive final quark

The first order corrections to the inclusive cross section have been computed in [18] for the charm production in charged-current DIS,

$$\nu_\mu+s\to\mu+c+(g), \tag{88}$$

[11] Let us note that logarithmic mass effects only occur for $rN\gg1$ because, for $rN\ll1$, one can expand the exponent in (85) as

$$z^{r(N-1)}-1=r(N-1)\log z+\frac{1}{2}r^2(N-1)^2\log^2z+\cdots, \tag{86}$$

obtaining power-suppressed effects of the usual form. The use of the resummation formula in the latter case however is not legitimate.

where (g) is a real or a virtual gluon. In this computation, the non-zero charm mass has been retained while the (much smaller) strange quark mass has been neglected:

$$m_c = m \neq 0; \qquad m_s = 0. \tag{89}$$

Omitting non-logarithmic terms, the cross section given in Eq. (40) of [18] can be written as:

$$\frac{1}{\sigma}\frac{d\sigma}{dx} \simeq \delta(1-x) - \frac{\alpha}{2\pi}\frac{1}{\bar{\epsilon}}P_{qq}^{(0)}(x) + \frac{\alpha}{2\pi}P_{qq}^{(0)}(x)\log\frac{Q^2}{\mu_F^2} + \frac{C_F\alpha}{\pi}\left[\frac{2\log(1-x)-\log(1-\lambda x)}{1-x}\right]_+ - \frac{C_F\alpha}{\pi}\frac{1}{[1-x]_+} + \frac{1}{4}\frac{C_F\alpha}{\pi}\frac{1}{[1-\lambda x]_+}, \tag{90}$$

where

$$\lambda \equiv \frac{Q^2}{Q^2+m^2}, \tag{91}$$

$x = x_B \equiv Q^2/(2p\cdot q)$ is the Bjorken variable, $Q^2 \equiv -q^2$ is the hard scale squared, $q = p_\mu - p_{\nu_\mu}$ is the W-boson momentum, μ_F is the unit of mass of dimensional regularization (to become the factorization scale after pole subtraction), $P_{qq}^{(0)}(x)$ is the leading-order (massless) $q \to q$ splitting function in 4 dimensions,

$$P_{qq}^{(0)}(x) \equiv C_F\left[\frac{1+x^2}{1-x}\right]_+ \tag{92}$$

and

$$\frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \gamma_E + \log(4\pi), \tag{93}$$

with $d = 4-2\epsilon$ the space–time dimension. For simplicity's sake, let us set from now on $\mu_F = Q$. The infrared pole, of collinear nature, is absorbed into the quark non-singlet distribution function, which reads, in the $\overline{\text{MS}}$ scheme:

$$F_{q/q}\big(x;Q^2\big) = \delta(1-x) - \frac{\alpha}{2\pi}\frac{1}{\bar{\epsilon}}P_{qq}^{(0)}(x) + O\big(\alpha^2\big). \tag{94}$$

The next step is to factorize from the cross section in Eq. (90) the (massless) coefficient function:

$$C_{\text{DIS}}\big(x;Q^2\big) = \delta(1-x) + A_1\alpha\left[\frac{\log(1-x)}{1-x}\right]_+ + B_1\alpha\frac{1}{[1-x]_+} + O\big(\alpha^2\big). \tag{95}$$

The latter is obtained as the following convolution [2–4]

$$C_{\text{DIS}}\big(x;Q^2\big) = \Delta\big(x;Q^2\big)\otimes J\big(x;Q^2\big), \tag{96}$$

where

$$\Delta\big(x;Q^2\big) = \delta(1-x) + 2A_1\alpha\left[\frac{\log(1-x)}{1-x}\right]_+ + O\big(\alpha^2\big), \tag{97}$$

is the radiative factor related to the observed initial-state jet, produced by the massless s quark, while

$$J\big(x;Q^2\big) = \delta(1-x) - A_1\alpha\left[\frac{\log(1-x)}{1-x}\right]_+ + B_1\alpha\frac{1}{[1-x]_+} + O\big(\alpha^2\big) \tag{98}$$

is the jet factor related to the unobserved final-state jet, initiated by the c quark, in the massless approximation. The cross section can therefore be written as:

$$\frac{1}{\sigma}\frac{d\sigma}{dx} \simeq F\big(x;Q^2\big)\otimes\Delta\big(x;Q^2\big)\otimes J\big(x;Q^2\big)\otimes h\big(x;Q^2,m^2\big), \tag{99}$$

where:

$$h\big(x;Q^2,m^2\big) = \delta(1-x) + A_1\alpha\left\{\left[\frac{\log(1-x)}{1-x}\right]_+ - \left[\frac{\log(1-x+r)}{1-x}\right]_+\right\} + (D_1-B_1)\alpha\left\{\frac{1}{[1-x]_+} - \frac{1}{[1-x+r]_+}\right\}. \tag{100}$$

The "$\otimes$" denotes a convolution and we have used the fact that

$$\lambda = \frac{1}{1+r} \simeq 1 - r \tag{101}$$

and

$$1 - \lambda x \cong 1 - x + r. \tag{102}$$

The last term $h(x; Q^2, m^2)$ on the r.h.s. of Eq. (99) is in agreement, within logarithmic accuracy, with $\delta(y; Q^2, m^2)$ expanded to $O(\alpha)$, in Eq. (87), after setting $x = 1 - y$:

$$h\left(x; Q^2, m^2\right) \simeq \delta\left(1 - x; Q^2, m^2\right). \tag{103}$$

The resummation of mass effects in the charged-current DIS cross section, for several values of the ratio m_c/Q, has been investigated in detail in [19].

3.3. Tower expansion

The universal mass-correction factor has the generalized exponential structure [2]

$$\delta_N\left(Q^2; m^2\right) = e^{F_N(Q^2; m^2)}, \tag{104}$$

where the exponent has a double expansion of the form:

$$F_N\left(Q^2; m^2\right) = \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} F_{nk} \alpha^n \log^k(Nr), \tag{105}$$

with F_{nk} numerical coefficients. The exponent can be expanded in towers of logarithms as:

$$F_N\left(Q^2; m^2\right) = \rho d_1(\rho) + \sum_{n=0}^{\infty} \alpha^n d_{n+2}(\rho) = \rho d_1(\rho) + d_2(\rho) + \alpha d_3(\rho) + \alpha^2 d_4(\rho) + \cdots, \tag{106}$$

where

$$\rho \equiv \beta_0 \alpha\left(\mu^2\right) \log(Nr) \tag{107}$$

and $\mu = O(m)$ is a renormalization scale of the order of the quark mass m.

By truncating the above series expansion, one obtains a fixed-logarithmic approximation to the form factor δ_N. The functions $d_i(\rho)$, which represent the mass effects, can be obtained from the standard ones $g_i(\lambda)$ [12] by means of the replacements[12]:

$$A(\alpha) \to -A(\alpha); \qquad B(\alpha) \to -B(\alpha); \qquad D(\alpha) \to D(\alpha); \qquad \log\frac{\mu^2}{Q^2} \to \log\frac{\mu^2}{m^2}; \qquad \lambda \to \rho. \tag{108}$$

It is worth observing that mass effects induce a similar structure to the massless one, involving changes of sign of the collinear functions A and B, with the rescaling $Q \to m$. We then obtain:

$$d_1(\rho) = \frac{A_1}{2\beta_0 \rho}\left[(1-2\rho)\log(1-2\rho) - 2(1-\rho)\log(1-\rho)\right]; \tag{109}$$

$$\begin{aligned} d_2(\rho) = {} & \frac{D_1}{2\beta_0}\log(1-2\rho) - \frac{B_1}{\beta_0}\log(1-\rho) - \frac{A_2}{2\beta_0^2}\left[\log(1-2\rho) - 2\log(1-\rho)\right] \\ & + \frac{A_1\beta_1}{4\beta_0^3}\left[2\log(1-2\rho) + \log^2(1-2\rho) - 4\log(1-\rho) - 2\log^2(1-\rho)\right] \\ & - \frac{A_1\gamma_E}{\beta_0}\left[\log(1-2\rho) - \log(1-\rho)\right] - \frac{A_1}{2\beta_0}\left[\log(1-2\rho) - 2\log(1-\rho)\right]\log\frac{\mu^2}{m^2}. \end{aligned} \tag{110}$$

For the NNLO function d_3 we obtain:

[12] All these functions contain in principle the over-all factor $\theta(N - 1/r)$, coming from the step approximation of the moment kernel, which avoids modifications for small N of the massless behavior, in agreement with physical intuition. Analytic continuation to the complex N-plane is made by omitting such factor.

$$
\begin{aligned}
d_3(\rho) = {} & -\frac{D_2}{\beta_0}\frac{\rho}{1-2\rho} - D_1\gamma_E\frac{2\rho}{1-2\rho} + \frac{D_1\beta_1}{2\beta_0^2}\left[\frac{2\rho}{1-2\rho} + \frac{\log(1-2\rho)}{1-2\rho}\right] + \frac{B_2}{\beta_0}\frac{\rho}{1-\rho} + B_1\gamma_E\frac{\rho}{1-\rho} \\
& - \frac{B_1\beta_1}{\beta_0^2}\left[\frac{\rho}{1-\rho} + \frac{\log(1-\rho)}{1-\rho}\right] + \frac{A_3}{2\beta_0^2}\left[\frac{\rho}{1-2\rho} - \frac{\rho}{1-\rho}\right] + \frac{A_2\gamma_E}{\beta_0}\left[\frac{2\rho}{1-2\rho} - \frac{\rho}{1-\rho}\right] \\
& - \frac{A_2\beta_1}{2\beta_0^3}\left[\frac{3\rho}{1-2\rho} - \frac{3\rho}{1-\rho} + \frac{\log(1-2\rho)}{1-2\rho} - \frac{2\log(1-\rho)}{1-\rho}\right] \\
& + \frac{A_1\gamma_E^2}{2}\left[\frac{4\rho}{1-2\rho} - \frac{\rho}{1-\rho}\right] + \frac{A_1\pi^2}{12}\left[\frac{4\rho}{1-2\rho} - \frac{\rho}{1-\rho}\right] \\
& + \frac{A_1\beta_2}{4\beta_0^3}\left[\frac{2\rho}{1-2\rho} - \frac{2\rho}{1-\rho} + 2\log(1-2\rho) - 4\log(1-\rho)\right] \\
& - \frac{A_1\beta_1\gamma_E}{\beta_0^2}\left[\frac{2\rho}{1-2\rho} - \frac{\rho}{1-\rho} + \frac{\log(1-2\rho)}{1-2\rho} - \frac{\log(1-\rho)}{1-\rho}\right] \\
& + \frac{A_1\beta_1^2}{2\beta_0^4}\left[\frac{\rho}{1-2\rho} - \frac{\rho}{1-\rho} - \log(1-2\rho) + \frac{\log(1-2\rho)}{1-2\rho} + \frac{\log^2(1-2\rho)}{2(1-2\rho)}\right. \\
& \left. + 2\log(1-\rho) - \frac{2\log(1-\rho)}{1-\rho} - \frac{\log^2(1-\rho)}{1-\rho}\right] - \frac{D_1}{\beta_0}\frac{\rho}{1-2\rho}\log\frac{\mu^2}{m^2} \\
& + \frac{B_1}{\beta_0}\frac{\rho}{1-\rho}\log\frac{\mu^2}{m^2} + \frac{A_2}{\beta_0^2}\left[\frac{\rho}{1-2\rho} - \frac{\rho}{1-\rho}\right]\log\frac{\mu^2}{m^2} + \frac{A_1\gamma_E}{\beta_0}\left[\frac{2\rho}{1-2\rho} - \frac{\rho}{1-\rho}\right]\log\frac{\mu^2}{m^2} \\
& - \frac{A_1\beta_1}{\beta_0^3}\left[\frac{\rho}{1-2\rho} - \frac{\rho}{1-\rho} + \frac{\log(1-2\rho)}{2} + \frac{\rho\log(1-2\rho)}{1-2\rho} - \log(1-\rho) - \frac{\rho\log(1-\rho)}{1-\rho}\right]\log\frac{\mu^2}{m^2} \\
& + \frac{A_1}{2\beta_0}\left[\frac{2\rho^2}{1-2\rho} - \frac{\rho^2}{1-\rho}\right]\log^2\frac{\mu^2}{m^2}.
\end{aligned}
\tag{111}
$$

The coefficients β_i of the QCD β-function in our normalization have been given in [12].

3.4. Inverse Mellin transform

The mass-correction factor in physical space is obtained by means of an inverse Mellin transform of δ_N:

$$
\delta\big(y; Q^2, m^2\big) = (1-y)\frac{d}{dy}\left\{\int_{c-i\infty}^{c+i\infty}\frac{dN}{2\pi i N}(1-y)^{-N}\delta_N\big(Q^2, m^2\big)\right\},
\tag{112}
$$

where c is a (real) constant chosen in such a way that the integration contour lies to the right of all the singularities of δ_N. By defining

$$
\bar{\delta}_{Nr} \equiv \delta_N
\tag{113}
$$

and changing variable from N to $\nu = Nr$, we obtain:

$$
\delta\big(y; Q^2, m^2\big) = \frac{d}{dy}\left\{\int_{cr-i\infty}^{cr+i\infty}\frac{d\nu}{2\pi i\nu}\big[(1-y)^{1/r}\big]^{-\nu}\bar{\delta}_\nu\big(Q^2, m^2\big)\right\},
\tag{114}
$$

where we have omitted the power correction y multiplying the derivative with respect to y. The correction factor in physical space is therefore the inverse Mellin transform of $\bar{\delta}_\nu$ with respect to $(1-y)^{1/r}$.[13] We can therefore use the results in [17] to obtain the correction factor in physical space in NNLL approximation:

$$
\delta\big(y; Q^2, m^2\big) = \frac{d}{dy}\big\{\theta(y)\Delta\big(y; Q^2, m^2\big)\big\},
\tag{115}
$$

where:

$$
\Delta\big(y; Q^2, m^2\big) = \frac{e^{Ld_1(\tau)+d_2(\tau)}}{\Gamma[1-h_1(\tau)]}\delta\Delta.
\tag{116}
$$

[13] Let us note that, in the large N limit we are interested in (see Eq. (82)) $rN \gg 1$, we can make the approximation $r(N-1) \simeq rN - 1$.

We have defined

$$h_1(\tau) = \frac{d}{d\tau}\big[\tau d_1(\tau)\big] = d_1(\tau) + \tau d_1'(\tau). \tag{117}$$

$\delta\Delta$ is an NNLL correction factor which can be set equal to one in NLL:

$$\delta\Delta_{\mathrm{NLL}} = 1. \tag{118}$$

Its NNLL expression reads:

$$\delta\Delta = \frac{S}{S|_{L\to 0}} \tag{119}$$

with

$$S = e^{\alpha d_3(\tau)}\left\{1 + \beta_0\alpha d_2'(\tau)\psi\big[1-h_1(\tau)\big] + \frac{1}{2}\beta_0\alpha h_1'(\tau)\big\{\psi^2\big[1-d_1(\tau)\big] - \psi'\big[1-d_1(\tau)\big]\big\}\right\}. \tag{120}$$

$\Gamma(x)$ is the Euler Gamma function and

$$\psi(x) \equiv \frac{d}{dx}\log\Gamma(x) \tag{121}$$

is the digamma function. Finally:

$$\tau \equiv \beta_0\alpha L \tag{122}$$

and

$$L \equiv -\log\big[1-(1-y)^{1/r}\big]. \tag{123}$$

It is convenient to approximate the argument of the logarithm by an expansion for $y \ll r$:

$$(1-y)^{1/r} = 1 - \frac{y}{r} + O\left(\frac{y^2}{r^2}\right). \tag{124}$$

Note that the r.h.s. is positive only for $y < r$, implying that the linearization above shrinks the domain of y from the unitary interval $(0,1)$ to the much smaller interval $(0,r)$.

To summarize, we have the final result:

$$\delta\big(y;Q^2,m^2\big) = \frac{d}{dy}\big\{\theta(y)\Delta\big(y;Q^2,m^2\big)\big\}, \tag{125}$$

where $\Delta(y;Q^2,m^2)$ is given in Eq. (116) and

$$L = \theta(r-y)\log\frac{r}{y}. \tag{126}$$

In agreement with the observation above, we have limited the domain to $y < r$ with a θ-function.

3.4.1. Coefficients of the mass logarithms

The mass correction factor in physical space $\Delta(y;Q^2,m^2)$ also has a generalized exponential structure:

$$\Delta\big(y;Q^2,m^2\big) = e^{H(y;Q^2,m^2)}, \tag{127}$$

where the exponent has a double expansion of the form:

$$H\big(y;Q^2,m^2\big) = \theta(r-y)\sum_{n=1}^{\infty}\sum_{k=1}^{n+1} H_{nk}\alpha^n\log^k\frac{r}{y}, \tag{128}$$

with H_{nk} numerical coefficients. By expanding the r.h.s. of Eq. (116) up to third order, one obtains the following expressions for the logarithmic coefficients:

$$H_{12}=\frac{1}{2}A_1; \tag{129}$$

$$H_{11}=B_1-D_1; \tag{130}$$

$$H_{23}=\frac{1}{2}A_1\beta_0; \tag{131}$$

$$H_{22}=\frac{1}{2}A_2+\frac{1}{2}\beta_0(B_1-2D_1)-\frac{1}{2}A_1^2z(2); \tag{132}$$

$$H_{21}=B_2-D_2-A_1(B_1-D_1)z(2)-A_1^2z(3); \tag{133}$$

$$H_{34}=\frac{7}{12}A_1\beta_0^2; \tag{134}$$

$$H_{33}=A_2\beta_0+\frac{1}{2}A_1\beta_1+\frac{1}{3}\beta_0^2(B_1-4D_1)-\frac{3}{2}A_1^2\beta_0z(2)-\frac{1}{3}A_1^3z(3); \tag{135}$$

$$H_{32}=\frac{1}{2}A_3+\beta_0(B_2-2D_2)+\frac{\beta_1}{2}(B_1-2D_1)-A_1A_2z(2)-\frac{A_1\beta_0}{2}(5B_1-7D_1)z(2)-\frac{A_1^3}{4}z(4)-\frac{9A_1^2\beta_0z(3)}{2}$$
$$-A_1^2(B_1-D_1)z(3), \tag{136}$$

where $z(3)=1.20206\ldots$ and $z(4)=\pi^4/90=1.08232\ldots$. Note that the leading coefficients H_{23} and H_{34} involve products of the one-loop coefficients A_1 and β_0 only. The explicit expressions of the coefficients read:

$$H_{12}=\frac{C_F}{2\pi}; \tag{137}$$

$$H_{11}=\frac{C_F}{4\pi}; \tag{138}$$

$$H_{23}=\frac{C_F}{4\pi^2}\left(\frac{11C_A}{6}-\frac{n_f}{3}\right); \tag{139}$$

$$H_{22}=-\frac{C_F^2z(2)}{2\pi^2}+\frac{C_F}{4\pi^2}\left[C_A\left(\frac{433}{72}-z(2)\right)-\frac{35n_f}{36}\right]; \tag{140}$$

$$H_{21}=\frac{C_F^2}{2\pi^2}\left(+z(2)-\frac{3}{16}-5z(3)\right)+\frac{C_F}{4\pi^2}\left[n_f\left(\frac{239}{108}-\frac{2z(2)}{3}\right)+C_A\left(\frac{-3595}{216}+\frac{5z(2)}{3}+19z(3)\right)\right]; \tag{141}$$

$$H_{34}=\frac{C_F}{48\pi^3}\left[C_A\left(\frac{847C_A}{36}-\frac{77n_f}{9}\right)+\frac{7n_f^2}{9}\right]; \tag{142}$$

$$H_{33}=-\frac{C_F^3z(3)}{3\pi^3}+\frac{C_F^2}{4\pi^3}\left[-\frac{11C_Az(2)}{2}+n_f\left(-\frac{1}{4}+z(2)\right)\right]+\frac{C_F}{4\pi^3}\left[\frac{11n_f^2}{36}+C_A^2\left(\frac{1711}{144}-\frac{11z(2)}{6}\right)\right.$$
$$\left.+C_An_f\left(-4+\frac{z(2)}{3}\right)\right]; \tag{143}$$

$$H_{32}=-\frac{C_F^3}{4\pi^3}\left[z(4)+z(3)\right]+\frac{C_F^2}{4\pi^3}\left[\frac{-67n_f}{48}+\frac{61n_fz(2)}{36}+5n_fz(3)+C_A\left(-\frac{11}{32}-\frac{767z(2)}{72}+2z(2)^2-22z(3)\right)\right]$$
$$+\frac{C_F}{4\pi^3}\left[\frac{-3n_f^2}{8}+\frac{n_f^2z(2)}{9}+C_A^2\left(\frac{-8855}{864}-\frac{145z(2)}{36}+\frac{11z(4)}{4}+\frac{319z(3)}{12}\right)+C_A\left(\frac{1549n_f}{432}-\frac{35n_fz(3)}{6}\right)\right], \tag{144}$$

where $C_A=N_c=3$ and n_f is the number of active quark flavors.

4. Conclusions

We have resummed the invariant mass distribution of hadronic jets initiated by massive quarks in next-to-leading logarithmic approximation. The resummation has later been extended in heuristic way to the next-to-next-to-leading logarithmic approximation. The mass effects have been relegated into a universal factor δ_N which takes into account the well-known dead-cone effect and soft radiation characteristic of massive partons. δ_N contains the same resummation functions which are encountered in standard threshold resummation. Mass corrections produce a universal intra-jet structure in agreement with one's physical intuition: only small angle partons emissions are involved, which can be ascribed to a specific jet in the event. It is remarkable that the coefficients of the mass logarithms are simply connected to those found in massless processes, at low order as well as in higher order of perturbation theory. A similar situation is found in the fragmentation of heavy quarks [4]. Our formulae have been checked against explicit first-order computations: the radiative b decay $b\to s\gamma$ with $m_s\neq 0$ and DIS $\nu_\mu+s\to c+\mu$ with $m_c\neq 0$, finding complete agreement.

Mass effects, i.e. effects related to non-vanishing parton masses, often play a significant role in jet physics at the quantitative level [20]. It may be worth to cite just a few applications of our results.

Semi-inclusive B-decays

$$B \to X_c + l + \nu_l \tag{145}$$

are largely affected by the non-vanishing charm quark mass as $r \gtrsim m_c^2/m_b^2 \approx 0.1$. Semileptonic $b \to c$ decays allow the extraction of the CKM matrix element V_{cb} and constitute the main background to the $b \to ul\nu$ decays, which are used for the extraction of the CKM matrix element V_{ub}. The inclusion of charm mass effects is needed to have a better understanding, for example, of the charged lepton spectrum or the invariant hadron mass distribution, which have recently been measured with great accuracy in [21–26]. The resummed formula which we have obtained can be combined with the full $O(\alpha_S)$ triple differential distribution for (145) recently obtained in [27,28]. An additional complication in this case stems from the fact that the charm quark velocity, or equivalently the parameter r, is not fixed in the tree-level process $b \to cl\nu$. That is because the hard scale, set by the total hadron energy in the final state, is not fixed. Since the exclusive channels $B \to Dl\nu$ and $B \to D^*l\nu$ constitute a large fraction of the total rate (145), an O(50%) [29], the use of perturbation theory can be questioned. We believe however that a perturbative computation can provide quantitative informations on the decay (145) to be compared with other models.

An accurate computation of shape variable distributions in e^+e^- annihilations at the Z^0 pole and below—such as thrust, heavy jet mass, C-parameter, etc.—asks for the inclusion of the beauty mass effects very close to the two-jet region. The mass correction parameter $r \approx 4m_b^2/s \approx 0.1$ for $\sqrt{s} = 30$ GeV, while it is smaller by an order of magnitude at the Z^0 pole. At a future e^+e^- linear collider of center-of-mass energy of 500 GeV, mass effects in top pair production will be controlled by $r \approx 0.5$.

The invariant hadron mass distribution in semileptonic top decays,

$$t \to X_b + W \tag{146}$$

is affected by the non-vanishing beauty mass close to threshold, i.e. for $m_{Xb} \gtrsim m_b$.

In Ref. [30] a disagreement has been found in the (massless) evolution of charm fragmentation data from 10 to 91 GeV. We argue that the inclusion of charm mass effects, which should be significant for $N \gtrsim (m_c/5)^{-2} \approx 10$, could improve the accuracy of the perturbative computations and eventually solve this problem.

To sum up, mass effects in a jet can be included, within logarithmic accuracy, by the universal factor in Eq. (18) multiplying the massless jet function.

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